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The space of super light rays for $D = 10$, $N = 1$ superconformal structures

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Abstract

We construct the space of super light rays for a $D = 10$, $N = 1$ superconformal structure. A super Ward correspondence is then established between $D = 10$, $N = 1$ supersymmetric Yang–Mills field equations on 10-dimensional spaces of the form $M_4 \times \mathbf{M}_6$ (where M_4 is a four-dimensional complex space–time and $\mathbf{M}_6$ is six-dimensional complex Minkowski space) and superbundles over the space of super light rays which are trivial on normal embedded quadrics. This result reduces in four dimensions to the equivalence of connections satisfying the $D = 4$, $N = 4$ SSYM field equations and superconnections integrable along super light rays also satisfying an added geometrical constraint coming from dimensional reduction. This extends the work of Witten and of Harnad and Shnider done on flat Minkowski space.

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1. $D = 10$, $N = 1$ superconformal structures

A superconformal structure on a 10|16-dimensional supermanifold, $M^{10|16}$, is defined by the existence of 0|16-dimensional distribution, T_1M such that the Frobenius form satisfies

$$[s_A, s_B]/T_1M = \Gamma^c_{AB} e_c$$

for some basis s_A of T_1M and e_c of $T_0M = TM/T_1M$ and where the Γ^c_{AB} are the 10-dimensional gamma matrices.

Locally, we may write a basis of T_1M in the form

$$q_A = \frac{\partial}{\partial\theta^A} + X^b_A \frac{\partial}{\partial x^b},$$

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where (x^a, θ^A) are super coordinates on M. Our definition requires the existence of isomorphisms g_A^B and h_b^a such that

$$[g_A^C q_C, g_B^D q_D] = \Gamma_{AB}^c h_c^b \frac{\partial}{\partial x^b} \quad \mathrm{mod} \quad T_1 M.$$

2. The space of super light rays

Let η^{ab} denote the constant diagonal tensor, i.e. what the metric is for complex Minkowski space. To construct the space of super light rays for our 10|16-dimensional superconformal structure, we consider

$$\Sigma = \{v_c \in T_0^* M | \eta^{ab} h_a^c v_c h_b^d v_d = 0\},$$

and the presymplectic form

$$\begin{aligned} \mathrm{d}(\xi_a \omega^a) &= \mathrm{d}\xi_a \wedge \omega^a + \xi_a \,\mathrm{d}\theta^A \wedge \mathrm{d}X_A^a = \mathrm{d}\xi_a \wedge \omega^a - \xi_a \,\mathrm{d}\theta^A \wedge \omega^b \partial_b X_A^a \\ &\quad - \xi_a \,\mathrm{d}\theta^A \wedge \mathrm{d}\theta^B (\tfrac{1}{2}(q_A X_B^a + q_B X_A^a)), \end{aligned}$$

where $\omega^a = \mathrm{d}x^a + \mathrm{d}\theta^A X_A^a$ and $\mathrm{d}\theta^A$ form a moving frame which is dual to the frame $\partial/\partial x^a$ and q_A. The kernel of the presymplectic form restricted to Σ is spanned by the set Q^A, $D_a = [Q^B, Q^C]\Gamma_{aBC}$, where

$$Q^A = \xi_a h_c^a \Gamma^{cAB} g_B^C \left(q_C - \xi_d \partial_b X_C^d \frac{\partial}{\partial \xi_b} \right).$$

Note that this kernel has dimension 1|8 since the rank of $h_a^b \xi_b \Gamma^{aAB}$ is 8.

This kernel is tangent to Σ since

$$\begin{aligned} Q^A(\eta^{ab} h_a^c \xi_c h_b^d \xi_d) &= 2\xi_e h_d^e \Gamma^{dAB} g_B^C (\eta^{ab}(q^C h_a^c)\xi_c h_b^d \xi_d - \xi_f \partial_c X_C^f \eta^{ab} h_a^c h_b^d \xi_d) \\ &= 2\xi_e h_d^e \Gamma^{dAB} \eta^{ab} (g_B^C (q_C h_a^c - (\partial_f X_C^c) h_a^f) h_b^d \xi_c \xi_d \\ &= 2\zeta_d \Gamma^{dAB} \eta^{ab} T_{Ba}^c h_c^{-1e} \zeta_b \zeta_e, \end{aligned}$$

where $\zeta_a = h_a^b \xi_b$.

$$\begin{aligned} Q^A(\eta^{ab} h_a^c \xi_c h_b^d \xi_d) &= 2\zeta_d \Gamma^{dAB} \eta^{ab} \Gamma_{aBE} t^{cE} h_c^{-1e} \zeta_b \zeta_e \\ &= 2(\Gamma^{dAB} \Gamma_{BE}^b + \Gamma^{bAB} \Gamma_{BE}^d) t^{cE} h_c^{-1e} \zeta_e \zeta_b \zeta_d \\ &= 2\eta^{bd} \delta_E^A t^{eE} \zeta_e \zeta_b \zeta_d = 2t^{eA} \zeta_e \eta^{bd} \zeta_b \zeta_d = 0, \end{aligned}$$

and where we have used the fact that $T_{Ba}^c = \{g_B^D q_D, h_a^d \partial_d\}_c = \Gamma_{aBE} t^{cE}$. This can be shown using the Bianchi identity [1]:

$$\Gamma_{AB}^a T_{Ca}^b + \Gamma_{CA}^a T_{Ba}^b + \Gamma_{BC}^a T_{Aa}^b = 0,$$

where

$$\Gamma_{AB}^a T_{Ca}^b = \Gamma_{AB}^a [g_C^D q_D, h_a^b \partial_b] = [g_C^D q_D, [g_A^E q_E, g_B^F q_F]] = [\mathbf{q}_C, [\mathbf{q}_A, \mathbf{q}_B]]$$

for $\mathbf{q}_C = g_C^D q_D$. Thus,

$$\begin{aligned} T^b_{Ac} &= -\tfrac{1}{16}\Gamma_c^{BC}(\Gamma^a_{CA}T^b_{Ba} + \Gamma^a_{AB}T^b_{Ca}) = -\tfrac{1}{8}\Gamma^a_{CA}(\Gamma_c^{BC}T^b_{Ba}) \\ &= -\tfrac{1}{8}(-\Gamma_{cCA}\Gamma^{aBC}T^b_{Ba} + 2\delta^a_c T^b_{Aa}), \end{aligned}$$

where we have used $\Gamma_c^{BC}\Gamma^a_{BC} = 16\delta^a_c$ and

$$\Gamma^{aAB}\Gamma^b_{BC} + \Gamma^{bAB}\Gamma^a_{BC} = 2\eta^{ab}\delta^A_C.$$

Thus,

$$8T^b_{AC} = \Gamma_{cCA}\Gamma^{aBC}T^b_{Ba} - 2T^b_{Ac},$$

and $T^b_{Ac} = (1/10)\Gamma_{cCA}t^{Cb}$.

Let F be the leaf space of this distribution. (It is integrable since it is the kernel of a closed 2-form.) As in [3], there is a $\mathbf{C}_*$-action on Σ given by scalar multiplication:

$$m_t : (x^a, \theta^A, \xi_b) \mapsto (x^a, \theta^A, t\xi_b).$$

We have $m_t^*\phi = t\phi$, so for $v \in \ker\phi$,

$$\phi \sqcup m_t^* v = m_t^*\phi \sqcup v = t\phi \sqcup v = 0,$$

where $\sqcup$ represents contraction. m_{t*} is clearly injective so $m_{t*}\ker\phi = \ker\phi$ and leaves are taken onto leaves by m_t. We define the space of super light rays, N, as $N = F/\mathbf{C}_*$. It also appears that we may perhaps go and define a supercontact structure on N as in [3,6], but we shall not do so here.

3. The embedding of $d = 4$, $N = 4$ superconformal structures

Let us first recall the definition of a superconformal structure on a $4|4N$-dimensional supermanifold [6]. It is defined by the existence of supervector bundles $S_+^{2|0}$, $S_-^{2|0}$, $E^{0|N}$ and the exact sequence

$$0 \to T_l M \oplus T_r M \to TM \to TM_0 \to 0,$$

where we have isomorphisms

$$T_l M \cong S_+ \otimes E, \qquad T_r M \cong S_- \otimes E^*, \qquad T_0 M \cong S_+ \otimes S_-.$$

$T_l M$ and $T_r M$ are required to be integrable distributions and the Frobenius form

$$\Phi : T_l M \otimes T_r M \to T_0 M,$$

where

$$\Phi(X \otimes Y) = [X, Y] \quad \mathrm{mod} \quad (T_l M \oplus T_r M)$$

is required to coincide via the above isomorphism with the convolution:

$$S_+ \otimes E \otimes E^* \otimes S_- \to S_+ \otimes S_-.$$

The Frobenius form is then said to be nondegenerate. (This definition is a simple generalization of the $N = 1$ case given in [5] which is based on the work of Ogievetskii and Sokachev [8].)

Let M_4 be a geodesically convex space–time. M_4 has a canonical $N = 4$ superconformal extension, $M_{4|16}$ [6]. We wish to embed $M_{4|16}$ as a factor of a $d = 10$, $N = 1$ superconformal structure. To do so we will need the following claim.

Claim. Let $M_{4|16}$ be the canonical superconformal extension of a space–time, M_4. The isomorphisms $g^{\beta j}_{\alpha i} : T_l M \to S_+ \otimes E$ and $g^{k\dot\beta}_{\dot\alpha l} : T_r M \to S_- \otimes E^*$ used in the definition of the superconformal structure on $M_{4|16}$, may each be taken to be the identity (see also [7]).

With this claim, it is straightforward to embed $M_{4|16}$ into a $d = 10$, $N = 1$ superconformal structure. Let $\Lambda = M_4 \times \mathbf{M}_6$, where $\mathbf{M}_6$ is six-dimensional Minkowski space, and define $T_1 \Lambda^{10|16}$ as the span of

$$\partial_{\alpha i} + X^{\bar a}_{\alpha i}\partial_{\bar a} + \epsilon_{\alpha\beta}\theta^{\beta k}\partial_{kl},$$

and

$$\partial^j_{\dot\alpha} + X^{\bar a j}_{\dot\alpha}\partial_{\bar a} + \epsilon^{jklm}\epsilon_{\dot\alpha\dot\beta}\theta^{\dot\beta}_k\partial_{lm},$$

where $\partial_{\alpha i} + X^{\bar a}_{\alpha i}\partial_{\bar a}$ span $T_l \Lambda^{10|16}$ and $\partial^j_{\dot\alpha} + X^{\bar a j}_{\dot\alpha}\partial_{\bar a}$ span $T_r \Lambda^{10|16}$.

To prove the claim, recall the double fibration

$$\begin{array}{ccc} & F_6 & \\ {}^{b}\swarrow & & \searrow^{a} \\ N_5 & & M_4 \end{array}$$

of reduced manifolds, and that $M_{4|4N}$ is defined as

$$\begin{aligned}
&(M_4, \mathcal{O}(\wedge^\bullet((a^0_*(\mathcal{O}(1,0)^{\oplus N} \oplus \mathcal{O}(0,1)^{\oplus N}))^*) \\
&\quad = (M_4, \mathcal{O}(\wedge^\bullet((a^0_*(\mathcal{O}(1,0)))^{*\oplus N} \oplus (a^0_*(\mathcal{O}(0,1)))^{*\oplus N}) \\
&\quad = (M_4, \mathcal{O}(\wedge^\bullet(S_+^{*\oplus N} \oplus S_-^{*\oplus N}))) = (M_4, \mathcal{O}(\wedge^\bullet(S_+^* \otimes E^* + S_-^* \otimes E)).
\end{aligned}$$

This identification is also compatible with the requirement that the Frobenius form be just the contraction,

$$S_+ \otimes E \otimes S_- \otimes E^* \to S_+ \otimes S_-.$$

To see this let $\theta^{\alpha i} = s^\alpha \otimes e^i$ and $\theta^{\dot\alpha}_j = s^{\dot\alpha} \otimes e_j$ and let $q_{\alpha i}, q^j_{\dot\alpha}$ be dual to $\mathrm{d}\theta^{\alpha i}, \mathrm{d}\theta^{\dot\alpha}_j$. Recall the exact sequence

$$0 \to H^0(Q \times U, \mathcal{O}(1,0) \otimes T) \oplus H^0(Q \times U, \mathcal{O}(0,1) \otimes T^*)$$
$$\to H^0(Q \times U, N) \to H^0(Q \times U, L) \to 0,$$

where $Q = \mathbf{P}_1 \times \mathbf{P}_1$ and U is an open coordinate patch in M.

Hence, we also have the exact sequence,

$$0 \to (H^0(L))^* \to (H^0(N))^* \to (H^0(\mathcal{O}(1,0) \otimes T))^* \oplus (H^0(\mathcal{O}(0,1) \otimes T^*))^* \to 0.$$

Now $[q_{\alpha i}, q^j_{\dot\alpha}]/T_1M = a_*[X + TQ, Y + TQ]/T_1M$, where $X, Y \in \Gamma(Q \times U, N)$ with

$$0 \to TQ \to b^*T\mathcal{N} \to N \to 0.$$

Thus $[X + TQ, Y + TQ] = b^*\,\mathrm{d}\theta(X + TQ, Y + TQ) = b^*\,\mathrm{d}\theta(X, Y)$, where θ is the contact structure on $\mathcal{N}$ and we have applied the fact that contraction of $\mathrm{d}\theta$ with an odd component of the normal bundle and a section of TQ is 0 and that contraction of $\mathrm{d}\theta$ with a section of TQ and a section of TQ is 0.

From the construction given in [6], we see that θ can be given as

$$\theta = \mathrm{d}q_0 + p^a\,\mathrm{d}q_a + \phi^j\,\mathrm{d}\psi_j,$$

and thus

$$\mathrm{d}\theta = \mathrm{d}p^a \wedge \mathrm{d}q_a + \mathrm{d}\phi^j \wedge \mathrm{d}\psi_j.$$

The important thing to note is that ϕ^j and ψ_j are local sections of $\mathcal{O}(-1,0) \otimes T$ and $\mathcal{O}(0,-1) \otimes T^*$ which locally can be written as $\phi^j = s_- \otimes e^j$ and $\psi_j = s_+ \otimes e_j$ and that if restricted to a normal embedded quadric, Q, and its odd normal bundle $\mathcal{O}(1,0) \otimes T^* \oplus \mathcal{O}(0,1) \otimes T$, the map

$$\mathrm{d}\theta|_Q : (\mathcal{O}(1,0) \otimes T^*) \otimes (\mathcal{O}(0,1) \otimes T) \to \mathcal{O}(1,1)$$

is just contraction of T and T^*. Thus on global sections

$$H^0(Q \times U, \mathcal{O}(1,0) \otimes T) \oplus H^0(Q \times U, \mathcal{O}(0,1) \otimes T^*) \to H^0(Q \times U, L),$$

we have

$$\mathrm{d}\theta(s_\alpha \otimes e_i, s_{\dot\alpha} \otimes e^j) = \delta^j_i s_\alpha \otimes s_{\dot\alpha},$$

and hence

$$[q_{\alpha i}, q^j_{\dot\alpha}]/T_1M = \delta^j_i h^{\beta\dot\beta}_{\alpha\dot\alpha} \partial_{\beta\dot\beta}/T_1M.$$

Thus, our odd coordinates are such that $g^{\beta j}_{\alpha i}$ and $g^{k\dot\beta}_{\dot\alpha l}$ are just the identity.

4. Integrability along super light rays and the field equations

Following [2,9], it is straightforward to show the equivalence of the field equations for the $d = 10$, $N = 1$ supersymmetric Yang–Mills theory and integrability of connections along the super light rays. We will thus have the following result:

Let $\Lambda = M_4 \times \mathbf{M}_6$. There is then a one-to-one correspondence between solutions to the field equations for $d = 10$, $N = 1$ supersymmetric Yang–Mills theory and connections which are integrable along super light rays.

We first show that a connection integrable along super light rays satisfies the superfield equations.

We repeat virtually verbatim the argument of Harnad and Shnider [2] as a matter of completeness:

For a connection given by

$$Q_A = q_A + \omega_A, \qquad D_a = h_a^b \partial_b + \omega_a,$$

integrability along super light rays, i.e. being able to solve for covariantly constant sections along super light rays,

$$\xi^a D_a \sigma = 0, \qquad \xi^a \Gamma_a^{AB} Q_B \sigma = 0$$

is determined by the constraint equation

$$[Q_A, Q_B] = \Gamma_{AB}^a D_a. \tag{1}$$

Define $\psi^B = (1/10)\Gamma^{aAB}[D_a, Q_A]$ and $F_{ab} = [D_a, D_b]$. Also define

$$\Sigma_B^{abA} = \tfrac{1}{2}(\Gamma^{aAC}\Gamma_{CB}^b - \Gamma^{bAC}\Gamma_{CB}^a).$$

It follows from the constraint equation and the Bianchi identities that the convariant derivative operator Q_A acts as follows on D_a, the Bosonic part of the connection, ψ_B and F_{ab}, respectively:

$$[Q_A, D_a] = -\Gamma_{aAB}\psi^B, \tag{2}$$

$$Q_A\psi^B = \tfrac{1}{2}\Sigma_A^{abB}F_{ab}, \tag{3}$$

$$Q_A F_{ab} = \Gamma_{aAB}D_b\psi^B - \Gamma_{bAB}D_a\psi^B. \tag{4}$$

These relations are all derived by Witten [9]. They follow in an elementary way from the definitions with the help of a number of identities satisfied by the 10-dimensional Γ-matrices which are listed here:

$$\Gamma^{aAB} = \Gamma^{aBA}, \qquad \Gamma_{AB}^a = \Gamma_{BA}^a, \qquad \Gamma^{aAB}\Gamma_{BC}^b + \Gamma^{bAB}\Gamma_{BC}^a = 2\eta^{ab}\delta_C^A,$$

$$\Gamma_{AB}^a\Gamma_{aCD} + \Gamma_{AC}^a\Gamma_{aBD} + \Gamma_{AD}^a\Gamma_{aBC} = 0, \qquad \Gamma_{AB}^a\Gamma_a^{AC} = 10\delta_B^C, \qquad \Gamma_{AB}^a\Gamma_b^{AB} = 16\delta_b^a,$$

$$\Sigma_B^{abA}\Gamma_{bAC} = -9\Gamma_{BC}^a, \qquad \Sigma_B^{abA}\Gamma_b^{BC} = +9\Gamma^{aAC},$$

$$\Gamma_{AB}^c\Sigma_C^{abB} + \Gamma_{BC}^c\Sigma_A^{abB} = -2\eta^{cb}\Gamma_{AC}^a + 2\eta^{ca}\Gamma_{AC}^b,$$

$$\Sigma_B^{abA}\Gamma^{cBC} + \Sigma_B^{abC}\Gamma^{cBA} = 2\eta^{bc}\Gamma^{aAC} - 2\eta^{ac}\Gamma^{bAC},$$

$$\Gamma^{aCB}\Gamma_{aAD}\Gamma^{bED}\Gamma_{bFB} = -4\Gamma^{aCE}\Gamma_{aFA} + 12\delta_F^C\delta_A^E + 8\delta_A^C\delta_F^E.$$

The equation, $Q_A\psi^B = (1/2)\Sigma_A^{abB}F_{ab}$, can be derived using the fact that $Q_A\psi^B = \Sigma_A^{abB}\chi_{ab}$ for some χ_{ab}, which follows from the Bianchi identity involving Q_A, Q_B, D_a.

Not all the identities listed were used in deriving the above equations but they are all useful in what follows in [2].

Now writing

$$\Gamma^a_{AB}D_a = \tfrac{1}{2}(Q_AQ_B + Q_BQ_A),$$

and applying it to ψ^B, using the above equations for each derivation by Q_A or Q_B, and the identity $\Sigma_B^{abA}\Gamma_b^{BC} = +9\Gamma^{aAC}$, we find the superfield Dirac equation:

$$\Gamma^a_{AB}D_a\psi^B = 0.$$

Finally, applying $\Gamma_b^{AC}Q_C$ to this equation and using the first two of the above equations along with several of the identities we find the superfield Yang–Mills equations:

$$D^aF_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B] = 0.$$

Simple reduction (i.e. "setting the Fermionic coordinates to zero") gives the field equations.

This completes Harnad and Shnider's summary of Witten's derivation of the superfield equations from the constraints. We note that the argument has been applied to the non-flat case we are studying. We also follow Harnad and Shnider's derivation that solutions to the $d = 10$, $N = 1$ supersymmetric Yang–Mills field equations correspond to connections integrable along super light rays, with some modification for the non-flat case we are studying.

Define in local coordinates, the transverse Euler vector field, $\mathcal{D} = \theta^A\partial_A$, and $\hat{\mathcal{D}} = \theta^A q_A$. Note that the eigenfunctions of $\mathcal{D}$ are superfields that are homogeneous in the θ^A coordinates and that the eigenvalues are just the order of homogeneity. Also note that $\mathcal{D}$ differs from $\hat{D}$ by an operator, $U = \theta^A X_A^a\partial_a$, which when applied to a superfield, increases the order of nilpotency by 2. In what follows we wish to consider superconnections that satisfy:

$$\theta^A\omega_A = 0.$$

A gauge may always be chosen so that such a condition is satisfied.

Using this gauge condition and Eqs. (1)–(4) we find that $\hat{\mathcal{D}}$ acts on the superconnection, the spinor superfield ψ^B and curvature superfield F_{ab} in the following manner:

$$(1+\hat{\mathcal{D}})\omega_B = 2\theta^A\Gamma^a_{AB}\omega_a, \qquad \hat{\mathcal{D}}\omega_a = -\theta^A\Gamma_{aAB}\psi^B, \qquad \hat{\mathcal{D}}\psi^B = +\tfrac{1}{2}\theta^A\Sigma_A^{abB}F_{ab},$$
$$\hat{\mathcal{D}}F_{ab} = \theta^A\Gamma_{aAB}D_b\psi^B - \theta^A\Gamma_{bAB}D_a\psi^B.$$

and thus for $\mathcal{D}$,

$$(1+\mathcal{D})\omega_B = 2\theta^A\Gamma^a_{AB}\omega_a + U\omega_B, \quad \mathcal{D}\omega_a = -\theta^A\Gamma_{aAB}\psi^B + U\omega_a,$$
$$\mathcal{D}\psi^B = +\tfrac{1}{2}\theta^A\Sigma_A^{abB}F_{ab} + U\psi^B, \quad \mathcal{D}F_{ab} = \theta^A\Gamma_{aAB}D_b\psi^B - \theta^A\Gamma_{bAB}D_a\psi^B + UF_{ab}.$$

If we are given a solution to the $d = 10$, $N = 1$ SSYM field equations, i.e. fields (ω_{0a}, ψ_0^B), which do not involve Fermionic (odd) coordinates, then this last set of equations can serve to define superfields $(\omega_a, \omega_A, \psi^B)$ which have (ω_{0a}, ψ_0^B) as leading components.

Having generated superfields from a solution to the field equations we wish to show the superfields are solutions to the superfield equations. We proceed by induction on the homogeneity in the θ^A coordinates. We have given a solution to the field equations:

$$\Gamma^a_{AB} D_a \psi^B = 0,$$

and

$$D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B] = 0$$

to order 0. Assume that these equations are true to order n. We will show they are true to order $n+1$ by applying $\mathcal{D}$. First the super Dirac equation.

$$\mathcal{D}\Gamma^a_{AB} D_a \psi^B = \hat{\mathcal{D}}\Gamma^a_{AB} D_a \psi^B + U\Gamma^a_{AB} D_a \psi^B.$$

By the induction hypothesis we have

$$\mathcal{D}\Gamma^a_{AB} D_a \psi^B = \hat{\mathcal{D}}\Gamma^a_{AB} D_a \psi^B.$$

Using the equations for the action of $\hat{\mathcal{D}}$ on our superfields we have

$$\begin{aligned}\mathcal{D}\Gamma^a_{AB} D_a \psi^B &= -\theta^C \Gamma^a_{AB}\Gamma_{aCD}[\psi^D, \psi^B] + \tfrac{1}{2}\theta^C \Gamma^a_{AB}\Sigma^{cdB}_C D_a F_{cd} \\ &= \theta^C(-\Gamma^a_{AC} D^c F_{ca} + \tfrac{1}{2}\Gamma^a_{AB}\Sigma^{cdB}_C D_a F_{cd}) = 0,\end{aligned}$$

where in the second line we have used the induction hypothesis that the super Yang–Mills field equation is true to order n and in the last step we used the Gamma matrix identity,

$$\Gamma^c_{AB}\Sigma^{abB}_C + \Gamma^c_{BC}\Sigma^{abB}_A = -2\eta^{cb}\Gamma^a_{AC} + 2\eta^{ca}\Gamma^b_{AC},$$

and the Bianchi identity for F_{cd}.

Similarly, we apply $\mathcal{D}$ to the super Yang–Mills equation to order $n+1$.

$$\begin{aligned}\mathcal{D}(D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B]) &= \hat{\mathcal{D}}(D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B]) \\ &\quad + U(D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B]).\end{aligned}$$

By the induction hypothesis we have

$$\mathcal{D}(D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B]) = \hat{\mathcal{D}}(D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B]).$$

Proceeding as before, using the $\hat{\mathcal{D}}$-equations and the induction hypothesis we have

$$\begin{aligned}&\mathcal{D}(D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^A, \psi^B]) \\ &\quad = -\Gamma^a_{CB}\theta^C[\psi^B, F_{ab}] + \theta^C D^a(\Gamma_{aCB} D_b \psi^B - \Gamma_{bCB} D_a \psi^B) \\ &\qquad + \tfrac{1}{2}\theta^C \Gamma_{bAB}\Sigma^{cdA}_C[F_{cd}, \psi^B] = -\Gamma^a_{CB}\theta^C([\theta^B, F_{ab}] + \Gamma^a_{CB}[F_{ab}, \psi^B] \\ &\qquad + \tfrac{1}{2}\Gamma_{bCA}\Sigma^{cdA}_B[F_{cd}, \psi^B] + \tfrac{1}{2}\Gamma_{bAB}\Sigma^{cdA}_C[F_{cd}, \psi^B]) = 0.\end{aligned}$$

In the induction step we have used the super Dirac equation (to order n) and its consequence:

$$D^a D_a \psi^B = -\tfrac{1}{2}\Sigma^{abB}_C[F_{ab}, \psi^C].$$

This completes Harnad and Shnider's inductive proof (with our modification, namely the use of $\hat{\mathcal{D}}$), that the solutions to the field equations generate solutions to the superfield equations. We now follow Harnad and Shnider's derivation that the superfield equations imply Eqs. (2)–(4).

Again using induction on the order of an expansion in the θ^A-coordinates, we first consider the zeroth-order:

$$[Q_A, D_a]_0 = \left(\frac{\partial}{\partial\theta^A}(\omega_a)\right)_0.$$

Since

$$\mathcal{D}\omega_a = (\hat{\mathcal{D}} - U)\omega_a,$$

where we recall that U is of order 2, we have

$$\omega_a = -\theta^D \Gamma_{aDC}\psi_0^C$$

to terms of order 1. Hence,

$$[Q_A, D_a]_0 = -\Gamma_{aAC}\psi_0^C.$$

Similarly,

$$[Q_A, \psi^B]_0 = \frac{\partial}{\partial\theta^A}\left(\frac{1}{2}\theta^C \Sigma_C^{abB} F_{0ab}\right) = \frac{1}{2}\Sigma_A^{abB} F_{0ab}.$$

Thus, Eqs. (2) and (3) hold at the zeroth-order. (Eq. (4) follows from Eq. (2) by taking even covariant derivatives.)

We now assume Eqs. (2) and (3) to be true to order n in θ^A and apply $1+\mathcal{D}$ to Eqs. (2) and (3) at order $n+1$. First, Eq. (2):

$$\begin{aligned}
(1+\mathcal{D})([Q_A, D_a] + \Gamma_{aAB}\psi^B) &= (1+\hat{\mathcal{D}}+U)([Q_A, D_a] + \Gamma_{aAB}\psi^B) \\
&= (1+\hat{\mathcal{D}})([Q_A, D_a] + \Gamma_{aAB}\psi^B) \\
&= (2\theta^C \Gamma_{AC}^b [D_b, D_a] + \theta^C Q_A(\Gamma_{aCB}\psi^B) + \tfrac{1}{2}\Gamma_{aAB}\theta^C \Sigma_C^{cdB} F_{cd} \\
&= \theta^C (2\Gamma_{AB}^b F_{ba} + \tfrac{1}{2}\Gamma_{aCB}\Sigma_A^{cdB} F_{cd} + \tfrac{1}{2}\Gamma_{aAB}\Sigma_C^{cdB} F_{cd}) = 0,
\end{aligned}$$

where we have used the induction hypothesis to order n and the Gamma identity:

$$\Gamma_{AB}^c \Sigma_C^{abB} + \Gamma_{BC}^c \Sigma_A^{abB} = -2\eta^{cb}\Gamma_{AC}^a + 2\eta^{ca}\Gamma_{AC}^b.$$

Applying $1+\mathcal{D}$ to both sides of Eq. (3), we have to order $n+1$,

$$\begin{aligned}
(1+\mathcal{D})(Q_A\psi^B - \tfrac{1}{2}\Sigma_A^{abB} F_{ab}) &= (1+\hat{\mathcal{D}}+U)(Q_A\psi^B - \tfrac{1}{2}\Sigma_A^{abB} F_{ab}) \\
&= (1+\hat{\mathcal{D}})(Q_A\psi^B - \tfrac{1}{2}\Sigma_A^{abB} F_{ab}) \\
&= 2\theta^C \Gamma_{AC}^a D_a\psi^B - \tfrac{1}{2}\theta^C \Sigma_C^{abB} Q_A F_{ab} - \Sigma_A^{abB}\theta^C \Gamma_{aCD} D_b\psi^B \\
&= 2\theta^C (\Gamma_{AC}^a D_a\psi^B - \Sigma_C^{abB}\Gamma_{aAD} D_b\psi^D - \Sigma_A^{abB}\Gamma_{aCD} D_b\psi^D) \\
&= 2\theta^C \Gamma_{AC}^a D_a\psi^B + \theta^C (\Gamma^{bBE}\Gamma_{EA}^a \Gamma_{aCD} + \Gamma^{bBE}\Gamma_{EC}^a \Gamma_{aAD}) D_b\psi^D \\
&= 2\theta^C \Gamma_{AC}^a D_a\psi^D - \theta^C (\Gamma^{bBE}\Gamma_{ED}^a \Gamma_{aAC}) D_b\psi^D = 0,
\end{aligned}$$

where we have used the Gamma identities

$$\Gamma^a_{AB}\Gamma_{aCD} + \Gamma^a_{AC}\Gamma_{aBD} + \Gamma^a_{AD}\Gamma_{aBC} = 0, \qquad \Gamma^{aAB}\Gamma^b_{BC} + \Gamma^{bAB}\Gamma^a_{BC} = 2\eta^{ab}\delta^A_C,$$

and the super Dirac equation,

$$\Gamma^a_{AB}D_a\psi^B = 0.$$

We now show that the superconnection constructed from the $\mathcal{D}$-recursion relations satisfies the constraint for integrability along super light rays, Eq. (1), by applying $(2+\hat{\mathcal{D}})$:

$$\begin{aligned}&(2+\hat{\mathcal{D}})([Q_A, Q_B] - 2\Gamma^a_{AB}D_a)\\ &\quad = 2[Q_A, Q_B] + [2\theta^C\Gamma^a_{AC}D_a, Q_B] + [Q_A, 2\theta^C\Gamma^a_{BC}D_a] - [Q_A, Q_B] - 4\Gamma^a_{AB}D_a\\ &\qquad + 2\Gamma^a_{AB}\theta^C\Gamma_{aCD}\psi^D = 2\theta^C\Gamma^a_{AC}\Gamma_{bBD}\psi^C + 2\theta^C\Gamma^a_{BC}\Gamma_{aAD}\psi^D\\ &\qquad + 2\theta^C\Gamma^a_{AB}\Gamma_{bCD}\psi^D = 0,\end{aligned}$$

where we have used Eq. (2) and the Gamma identity,

$$\Gamma^a_{AB}\Gamma_{aCD} + \Gamma^a_{AC}\Gamma_{aBD} + \Gamma^a_{AD}\Gamma_{aBC} = 0.$$

Thus Eq. (1) holds. *This completes the demonstration of Harnad and Shnider* [2]

Dimensional reduction to four dimensions produces as a corollary the equivalence of the field equations for the $d = 4$, $N = 4$ Yang–Mills theory and integrability along super light rays (in 4|16 dimensions) with the added constraint

$$\epsilon^{\alpha\beta}[Q_{\alpha i}, Q_{\beta j}] = \tfrac{1}{2}\epsilon_{ijkl}\epsilon^{\dot\alpha\dot\beta}[Q^k_{\dot\alpha}, Q^l_{\dot\beta}]$$

on our connection, $(D_{\alpha\dot\alpha}, Q_{\beta j}, Q^k_{\dot\beta})$.

Again we shall follow Harnad and Shnider [2] *in deriving this.*

Recall that we are considering 10-dimensional spaces of the form $M_4 \times \mathbf{M}_6$, where M_4 is four-dimensional complex conformal manifold and $\mathbf{M}_6$ is six-dimensional Minkowski space. Local coordinates are given by $(x^{\alpha\dot\alpha}, y^{ij})$, where $\alpha, \dot\alpha = 0, 1$, $i, j = 1, 2, 3, 4$ and y^{ij} is anti-symmetric in i and j. Given this decomposition, we can write a 10-dimensional spinor as

$$s_A = (s_{\alpha i}, s^j_{\dot\alpha}),$$

and a connection as

$$Q_{\alpha i} = q_{\alpha i} + \omega_{\alpha i}, \qquad Q^j_{\dot\alpha} = q^j_{\dot\alpha} + \omega^j_{\dot\alpha}, \qquad D_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + A_{\alpha\dot\alpha}, \qquad D_{ij} = \partial_{ij} + W_{ij}.$$

The constraint equation $[Q_A, Q_B] = \Gamma^a_{AB}D_a$, becomes

$$[Q_{\alpha i}, Q^j_{\dot\alpha}] = 2\delta^j_i D_{\alpha\dot\alpha}, \qquad [Q_{\alpha i}, Q_{\beta j}] = 2\epsilon_{\alpha\beta}D_{ij}, \qquad [Q^i_{\dot\alpha}, Q^j_{\dot\beta}] = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}D_{kl}.$$

For the spinor superfield ψ^B, define (using this decomposition):

$$\chi_{\dot\alpha i} = \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta}_i, \qquad \chi^i_\alpha = \tfrac{1}{2}\epsilon_{\alpha\beta}\psi^{\beta i}.$$

Also, define

$$[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}, \qquad [D_{\alpha\dot{\alpha}}, D_{ij}] = F_{\alpha\dot{\alpha},ij}, \qquad [D_{ij}, D_{kl}] = F_{ij,kl}.$$

The $d = 10$, $N = 1$ super Dirac field equation

$$\Gamma^a_{AB} D_a \psi^B = 0$$

in this notation becomes

$$\epsilon^{\alpha\beta} D_{\alpha\dot{\beta}} \chi^i_\beta + \tfrac{1}{2}\epsilon^{ijkl} D_{kl} \chi_{\dot{\beta}j} = 0, \qquad \epsilon^{\dot{\alpha}\dot{\beta}} D_{\alpha\dot{\alpha}} \chi_{\dot{\beta}i} + D_{ij} \chi^j_\alpha = 0,$$

and the $d = 10$, $N = 1$ super Yang–Mills equation,

$$D^a F_{ab} + \tfrac{1}{2}\Gamma_{bAB}[\psi^a, \psi^B] = 0$$

becomes

$$\epsilon^{\dot{\alpha}\dot{\beta}} D_{\gamma\dot{\alpha}} f_{\dot{\beta}\dot{\gamma}} + \epsilon^{\alpha\beta} D_{\alpha\dot{\gamma}} f_{\beta\gamma} + \tfrac{1}{4}\epsilon^{ijkl} D_{kl} F_{\gamma\dot{\gamma},ij} + [\chi^i_\gamma, \chi_{\dot{\gamma}i}] = 0,$$
$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} D_{\alpha\dot{\alpha}} F_{\beta\dot{\beta},ij} - \tfrac{1}{2}\epsilon^{mnkl} D_{mn} F_{kl,ij} + \tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijkl}[\chi^k_\alpha, \chi^l_\beta] + \epsilon^{\dot{\alpha}\dot{\beta}}[\chi_{\dot{\alpha}i}, \chi_{\dot{\beta}j}] = 0.$$

If we assume that our connection comes about as a pull back of a connection on our four-dimensional space, $M_{4|16}$, then our fields and superfields are also pull backs of fields and superfields on $M_{4|16}$. The action of the following operators on any of these fields or superfields is given by

$$D_{ij} \to [W_{ij},], \qquad F_{\alpha\dot{\beta},ij} \to D_{\alpha\dot{\beta}} W_{ij}, \qquad F_{ij,kl} \to [W_{ij}, W_{kl}].$$

The super Dirac equation then becomes:

$$\epsilon^{\alpha\beta} D_{\alpha\dot{\beta}} \chi^i_\beta + \tfrac{1}{2}\epsilon^{ijkl}[W_{kl}, \chi_{\dot{\beta}j}] = 0, \qquad \epsilon^{\dot{\alpha}\dot{\beta}} D_{\alpha\dot{\alpha}} \chi_{\dot{\beta}i} + [W_{ij}, \chi^j_\alpha] = 0,$$

and the super Yang–Mills equation becomes:

$$\epsilon^{\dot{\alpha}\dot{\beta}} D_{\gamma\dot{\alpha}} f_{\dot{\beta}\dot{\gamma}} + \epsilon^{\alpha\beta} D_{\alpha\dot{\gamma}} f_{\beta\gamma} + \tfrac{1}{4}\epsilon^{ijkl}[W_{kl}, D_{\gamma\dot{\gamma}} W_{ij}] + [\chi^i_\gamma, \chi_{\dot{\gamma}i}] = 0,$$
$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} D_{\alpha\dot{\alpha}} D_{\beta\dot{\beta}} W_{ij} - \tfrac{1}{2}\epsilon^{mnkl}[W_{mn}, [W_{kl}, W_{ij}]]$$
$$+ \tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijkl}[\chi^k_\alpha, \chi^l_\beta] + \epsilon^{\dot{\alpha}\dot{\beta}}[\chi_{\dot{\alpha}i}, \chi_{\dot{\beta}j}] = 0.$$

The constraint equation

$$[Q_{\alpha i}, Q^j_{\dot{\alpha}}] = \delta^j_i D_{\alpha\dot{\alpha}}$$

remains the same upon this dimensional reduction. The constraint equations

$$[Q_{\alpha i}, Q_{\beta j}] = 2\epsilon_{\alpha\beta} W_{ij}, \qquad [Q^i_{\dot{\alpha}}, Q^j_{\dot{\beta}}] = 2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{ijkl} W_{kl}$$

are equivalent to the equations

$$[Q_{\alpha i}, Q_{\beta j}] + [Q_{\beta i}, Q_{\alpha j}] = 0, \qquad [Q^i_{\dot{\alpha}}, Q^j_{\dot{\beta}}] + [Q^j_{\dot{\alpha}}, Q^i_{\dot{\beta}}] = 0,$$
$$\epsilon^{\alpha\beta}[Q_{\alpha i}, Q_{\beta j}] = \tfrac{1}{2}\epsilon_{ijkl}\epsilon^{\dot{\alpha}\dot{\beta}}[Q^k_{\dot{\alpha}}, Q^l_{\dot{\beta}}].$$

We may define $W_{ij} = \epsilon_{\alpha\beta}[Q_{\alpha i}, Q_{\beta j}]$ in order to show this equivalence. The last equation is a constraint that arises because of dimensional reduction from 10 to four dimensions, whereas the other three constraint equations are equivalent to integrability of our connection along super light rays in our space $M_{4|16}$.

5. The Ward correspondence

In this section we follow very closely along the lines of McHugh [6]. Let M_4 be a geodesically convex complex conformal space–time and let $\mathbf{M}_6$ be complex Minkowski space. Let $\Lambda = M_4 \times \mathbf{M}_6$. The space–time M_4 has a canonical extension to a superconformal manifold, $M_{4|16}$, which then can be embedded in the $D = 10$, $N = 1$ superconformal structure, $\Lambda_{10|16}$, which has a space of super light rays, $N_{17|8}$. We thus have the following double fibration of supermanifolds:

$$\begin{array}{ccc} & F_{18|16} & \\ & {\scriptstyle\rho}\swarrow \quad \searrow{\scriptstyle\pi} & \\ N_{17|8} & & \Lambda_{10|16} \end{array}.$$

Let Q_8, be the eight-dimensional quadric, i.e. the space of null directions in $\mathbf{C}^{10}$. For each $[v] \in Q_8$, let $E = co\ker(\Gamma_v : S_{16} \to S_{16})$, where Γ_v is the 10-dimensional gamma matrix contracted with v. E is an eight-dimensional bundle over Q_8. If Q is the image under ρ of a fiber of π then the normal bundle of Q in $N_{17|8}$ is $\nu_0 \oplus \nu_1$, where ν_0 is the normal bundle of Q in N_{rd17} and ν_1 is $(\mathcal{N}^2/\mathcal{N})|_Q$. By the results of Lebrun [3] $\nu_{0=}T\mathbf{P}_9|_{Q_8} \otimes \mathbf{H}^*$ and $\nu_1 = (T_1\Lambda_{10|16})/(\ker(\rho))_1 = S_{16}/\ker(\Gamma_v : S_{16} \to S_{16})$. On the other hand if Q is a quadric Q_8 in $N_{17|8}$ with normal bundle $T\mathbf{P}_9|_{Q_8} \otimes \mathbf{H}^* \oplus E$ then the normal bundle of Q in $N_{17|8}$ is $\nu_0 \oplus \nu_1$, where ν_0 is the normal bundle of Q in N_{rd17} and ν_1 is $(\mathcal{N}^2/\mathcal{N})|_Q$. Q is the image under ρ of a fiber of π. We thus have the following.

Lemma. *Let Q be the image under ρ of a fiber of π. The normal bundle of Q in $N_{17|8}$, is $T\mathbf{P}_9|_{Q_8} \otimes \mathbf{H}^* \oplus E$, where $\mathbf{H}$ is the canonical line bundle on $T\mathbf{P}_9$. Furthermore if Q is a quadric Q_8 in $N_{17|8}$ with normal bundle $T\mathbf{P}_9|_{Q_8} \otimes \mathbf{H}^* \oplus E$ then Q is the image under ρ of a fiber of π.*

We refer to such Q as normal embedded quadrics.

We shall follow Manin [5, pp. 73–74], and LeBrun [4, p. 1053] in showing the equivalence of connections with zero monodromy along any null line which are integrable along super light rays and vector bundles over the space of super light rays which are trivial when restricted to normal quadrics (see also [9]).

Assume the fibers of ρ, i.e. the super light rays of $\Lambda_{10|16}$, are connected. Let $(\mathcal{E}_\Lambda, \nabla)$ be a vector bundle with connection on Λ, which is integrable along super light rays and which has zero monodromy along these fibers. Let $TF/\mathcal{N} = \ker(\rho_*)$ and let $\nabla_{F/\mathcal{N}}$ be the composition

$$\pi^*\mathcal{E}_\Lambda \overset{\pi^*\nabla}{\to} \pi^*\mathcal{E}_\Lambda \otimes \pi^*\Omega^1\Lambda \overset{id\otimes\mathrm{res}}{\to} \Omega^1 F/\mathcal{N},$$

where res is the restriction to $TF/\mathcal{N}$. Define $\mathcal{E}'_F \equiv \ker(\nabla_{F/\mathcal{N}})$. Since $\nabla_{F/\mathcal{N}}$ has no curvature or monodromy and the fibers of ρ are connected, we have that $\mathcal{E}_{\mathcal{N}} = \rho_* \mathcal{E}'_F$ is a locally free sheaf on $\mathcal{N}$. Furthermore, this sheaf will be trivial when restricted to normal quadrics.

In order to show that a locally free sheaf on the space of super light rays, $\mathcal{N}$, which is trivial when restricted to normal quadrics gives rise to a vector bundle with connection on Λ, consider the short exact sequence of sheaves on F:

$$0 \to \mu \to \pi^* \Omega^1 \Lambda \to \Omega^1 F/\mathcal{N} \to 0,$$

where res is the map from $\pi^* \Omega^1 \Lambda$ to $\Omega^1 F/\mathcal{N}$ and μ is the kernel of that map. Over a small open set U in Λ we have the long exact sequence of $\mathcal{A}_{U \times Q_8}$ modules:

$$\begin{aligned} 0 &\to H^0(U \times Q_8, \mu) \to H^0(U \times Q_8, \pi^* \Omega^1 \Lambda) \to H^0(U \times Q_8, \Omega^1 F/\mathcal{N}) \\ &\to H^1(U \times Q_8, \mu) \to H^1(U \times Q_8, \pi^* \Omega^1 \Lambda) \to 0. \end{aligned}$$

Assuming $\Omega^1 \Lambda$ is trivial over U we have that the last term, $H^1(U \times Q_8, \pi^* \Omega^1 \Lambda)$ is 0.

We now show that res induces an injective map between $H^0(U \times Q_8, \pi^* \Omega^1 \Lambda)$ and $H^0(U \times Q_8, \Omega^1 F/\mathcal{N})$ and thus $H^0(U \times Q_8, \mu) = 0$. Let $\omega^a, \mathrm{d}\theta^A$ be a local frame of $H^0(U \times \mathbf{Q_8}, \pi^* \Omega^1 \Lambda)$ and let $\mathcal{Q}^A = \zeta_a \Gamma^{aAB} \mathbf{q}_B, \mathcal{D} = \zeta^a (\partial/\partial x^a) + \zeta^a \mathcal{R}_a^b (\partial/\partial \zeta^b)$ span $\ker(\mathrm{d}(\xi_a \omega^a) : TF \to \Omega^1 F)$. Here R_a^b are the coefficients of $\partial/\partial \zeta^b$ in $\mathcal{D}$ which we will not need to compute explicitly and

$$\mathbf{q}_B = g_B^C \left(q_C - \xi_d \partial_b X_C^d \frac{\partial}{\partial \xi_b} \right), \qquad \zeta_a = h_b^a \xi_b \quad \text{and} \quad \zeta^a = \eta^{ab} \zeta_b.$$

Let

$$\chi_a \omega^a + \lambda_A g_C^{-1A} \, \mathrm{d}\theta^C$$

be an arbitrary section of $H^0(U \times \mathbf{Q_8}, \pi^* \Omega^1 \Lambda)$. (Here λ_A is an arbitrary spinor.) We have

$$(\chi_a \omega^a + \lambda_A g_C^{-1A} \, \mathrm{d}\theta^C) \sqcup \mathcal{D} = \zeta^a \chi_a.$$

This will be zero for all null ζ^a, only if χ_a is zero.

Also

$$(\chi_a \omega^a + \lambda_A g_C^{-1A} \, \mathrm{d}\theta^C) \sqcup \mathcal{Q}^B = \lambda_B \zeta_a \Gamma^{aBA}.$$

If we choose ζ_a so that $\zeta_a \rho^a \neq 0$, where $\rho^a = \lambda_A \lambda_B \Gamma^{aAB}$ then $\lambda_B \zeta_a \Gamma^{aBA}$ will be zero only if $\lambda_B = 0$. Thus the the map, res, is injective and $H^0(U \times Q_8, \mu) = 0$.

We now have the short exact sequence

$$0 \to H^0(U \times Q_8, \pi^* \Omega^1 \Lambda) \to H^0(U \times Q_8, \Omega^1 F/\mathcal{N}) \to H^1(U \times Q_8, \mu) \to 0$$

of $\mathcal{A}_{U \times Q_8}$-modules and thus the short exact sequence of locally free sheaves

$$0 \to \Omega^1 \Lambda \to \pi_*^0 \Omega^1 F/\mathcal{N} \to \pi_*^1 \mu \to 0$$

on Λ. ($\mathcal{A}_{U \times Q_8}$ is the sheaf of superfunctions on $U \times Q_8$.)

We may consider a neighborhood of Λ, $\mathbf{U}$, in which this short exact sequence splits:

$$\pi_{*0}\Omega^1 F/\mathcal{N} = \Omega^1\Lambda \oplus \pi_*^1\mu,$$

and we thus have a projection map (over $\mathbf{U}$),

$$\text{proj} : \pi_*^0\Omega^1 F/\mathcal{N} \to \Omega^1\Lambda.$$

Given a super vector bundle, $\mathcal{E}_{\mathcal{N}}$, on the space of super light rays, $\mathcal{N}$, which is trivial when restricted to normal embedded quadrics, define $\mathcal{E}_\Lambda = \pi_*^0(\rho^*(\mathcal{E}_{\mathcal{N}}))$, we follow LeBrun [4] in defining a connection on $\mathcal{E}_{\mathbf{U}}$. Let

$$d_\rho : \mathcal{E}_{\mathbf{U}} \to \mathcal{E}_{\mathbf{U}} \otimes \pi_*^0(\Omega^1 F/\mathcal{N})$$

be differentiation along the fibers of ρ. We then consider the composition of this map with our projection map, proj,

$$\mathcal{E}_{\mathbf{U}} \to \mathcal{E}_{\mathbf{U}} \otimes \pi_*^0(\Omega^1 F/\mathcal{N}) \to \mathcal{E}_{\mathbf{U}} \otimes \Omega^1\mathbf{U}.$$

This is a superconnection which is integrable along the super light rays.

To obtain a Ward correspondence between Λ and its space of super light rays, we will piece together Ward transforms between open sets, $\mathbf{U}$ and the image of open sets $\mathbf{U}$ in $\mathcal{N}$. In the constructions given, a gauge transformation of the vector bundle and connection over the 10-dimensional space, $\mathbf{U}$, corresponds to a gauge transformation of the twistor transform over the space of super light rays. The same is true on the intersection of two such sets, namely, $\mathbf{U}_\alpha \cap \mathbf{U}_\beta$. We use the correspondence of gauge transformations on $\mathbf{U}_\alpha \cap \mathbf{U}_\beta$ to piece together Ward correspondences over each of the open sets $\mathbf{U}_\alpha$ to get a Ward correspondence over Λ.

Recall that a complex conformal space–time is said to be civilized if its space of null geodesics is a complex manifold. A complex conformal space–time is reflexive if it is the space of normal quadrics for its space of null geodesics. Let M_4 be a civilized and reflexive complex conformal space–time and let $\mathbf{M}_6$ be complex Minkowski space. We thus state the main theorem of this article:

Theorem. *Let $\Lambda = M_4 \times \mathbf{M}_6$, where M_4 is a complex conformal space–time and $\mathbf{M}_6$ is complex Minkowski space. There is then a one-to-one correspondence between*

1) *solutions to the field equations for $d = 10$, $N = 1$ supersymmetric Yang–Mills theory with no monodromy on any null line l, and*
2) *vector bundles over the space of super light rays, $N_{17|8}$, which are trivial when restricted to normal embedded quadrics, i.e. quadrics, $Q(p)$, which are the space of super light rays passing through a point, p in Λ.*

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References

[1] L.L. Chau, C.S. Lim, Geometrical constraints for $D = 10$, $N = 1$ supergravity, Int. J. Mod. Phys. 4 (15) (1989) 3813–3819.
[2] J. Harnad, S. Shnider, Constraints and field equations for ten-dimensional super Yang–Mills theory, Commun. Math. Phys. 106 (1986) 183–199.
[3] C.R. LeBrun, Space of complex null geodesics in complex Riemmanian geometry, Trans. AMS 278 (1983) 209–231.
[4] C.R. LeBrun, Thickenings and gauge fields, Class. Quant. Grav. 3 (1986) 1039–1059.
[5] Y. Manin, Gauge Fields and Complex Geometry (English Trans., Springer, Heidelberg, 1988), Nauka, Moskow, 1984.
[6] A.P. McHugh, The space of super light rays for complex conformal spacetimes, J. Geom. Phys. 13 (1994) 16–50.
[7] S.A. Merkulov, Quaternionic, quaternionic Kahler, and Hyper Kahler supermanifolds, Lett. Math. Phys. V25N1 (1992) 7–16.
[8] V.I. Ogievetskii, E.S. Sokachev, The simplest Einstein supersymmetry group, Sov. J. Nucl. Phys. 31 (1980) 140–148.
[9] E. Witten, Twistor-like transform in ten dimensions, Nucl. Phys. B 266 (1986) 245–264.